# BROWNIAN SHEET AND QUASI-SURE ANALYSIS

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*To Professor Miklos Csörgő on the occasion of his 70th birthday.*

Abstract. We present a self-contained and modern survey of some existing quasi-sure results via the connection to the Brownian sheet. Among other things, we prove that quasi-every continuous function: (i) satisfies the local law of the iterated logarithm; (ii) has Lévy's modulus of continuity for Brownian motion; (iii) is nowhere differentiable; and (iv) has a nontrivial quadratic variation. We also present a hint of how to extend (iii) to obtain a quasi-sure refinement of the M. Csörgő–P. Révész modulus of continuity for almost every continuous function along the lines suggested by M. Fukushima.

## 1. Introduction

Throughout, we let $\Omega$ denote the space of all continuous functions $f : [0, 1] \to \mathbb{R}$. As usual, $\Omega$ is endowed with the compact-open topology [i.e., the topology of uniform convergence] and its corresponding Borel sigma-algebra $\mathcal{B}(\Omega)$. Then a number of classical, as well as modern, theorems of probability theory can be interpreted a saying something about the "typical" function in $\Omega$ in the following sense:[1] If we endow $(\Omega, \mathcal{B}(\Omega))$ with the standard Wiener measure we then obtain the *classical Wiener space*, and various "probabilistic" results hold for almost every $f \in \Omega$. Here and throughout, "almost every" is tacitly understood to hold with respect to Wiener's measure on $(\Omega, \mathcal{B}(\Omega))$. As two notable examples we can consider the following, although frequently one thinks of these as statements about the Brownian motion:

(1) (Khintchine [27]). *Almost every $f \in \Omega$ satisfies the local law of the iterated logarithm; i.e.,*

$$\limsup_{x \to 0} \frac{f(x)}{\sqrt{2x \ln \ln \left(\frac{1}{x}\right)}} = -\liminf_{x \to 0} \frac{f(x)}{\sqrt{2x \ln \ln \left(\frac{1}{x}\right)}} = 1. \tag{1.1}$$

(2) (Paley, Wiener, and Zygmund [42]). *Almost every $f \in \Omega$ is nowhere-differentiable.*

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[1] The Baire category theorem provides us with another notion of "a typical function $f \in \Omega$," with a rich and colorful history dating back to the works of Baire, Borel, and Lebesgue. Among many gems of thought, J.-P. Kahane's article ([24, §3]) contains a delightful discussion of the history of this subject.

Other examples abound.

One might ask for a more restrictive notion of what it means for $f \in \Omega$ to be "typical." In the appendix to [15], D. Williams has proposed this problem, and has shown us an interesting, less restrictive, class of "typical functions" that is motivated by infinite-dimensional diffusion theory, and in particular the work of P. Malliavin in the said area ([37]).

Let $W$ denote the two-parameter Brownian sheet, based on which we can construct the *Ornstein–Uhlenbeck Brownian sheet*,

$$U(s,t) := e^{-s/2} W(e^s, t), \qquad \forall s,t \geq 0. \tag{1.2}$$

One can think of this two-parameter process as the "evaluations" of the following infinite-dimensional (in fact, $\Omega$-valued) stochastic process that is called the *Ornstein–Uhlenbeck process in Wiener space*:

$$Y_s := U(s, \bullet), \qquad \forall s \geq 0. \tag{1.3}$$

It is not difficult to see that $Y$ is an $\Omega$-valued diffusion; this follows at once from Theorem 2.1 below, for instance. Moreover, since $Y_0 = W(1, \bullet)$ is a standard Brownian motion, it follows that the process $Y$ is a stationary diffusion on the space of continuous function, and the invariant measure of $Y$ is Wiener's measure.

Next, consider the hitting probabilities $\mathrm{Cap}(\bullet)$ of the diffusion $Y$ killed at an independent mean-one exponential random variable; i.e., for any Borel set $G \subset \Omega$,

$$\mathrm{Cap}(G) := \int_0^\infty e^{-s} \mathrm{P}\left\{\exists s \geq 0 : \ Y_s \in G\right\} ds. \tag{1.4}$$

Following D. Williams, we then say that a Borel measurable set $G \subseteq \Omega$ holds *quasi-surely* if $\mathrm{Cap}(G^{\mathsf{c}}) = 0$.

It is not difficult to see that the set function Cap is a natural capacity in the sense of G. Choquet. From this it follows that $G$ holds quasi-surely if and only if its complement is a capacity-zero set; i.e., it is almost-surely never visited by the Ornstein–Uhlenbeck process on $\Omega$. Equivalently—and this requires only a moment of reflection—$G$ holds quasi-surely if and only if

$$\mathrm{P}\{\forall s \geq 0 : \ U(s, \bullet) \in G\} = 1. \tag{1.5}$$

Thanks to (1.2), the quasi-sure analysis of subsets of $\Omega$ can be related to the Brownian sheet.

An alternative, more direct, approach was proposed by M. Fukushima ([22]) who used the properties of the Dirichlet form associated with the infinite-dimensional process $Y$ to produce interesting quasi-sure theorems. This was an exciting new development on the intersection of probability and infinite-dimensional analysis, and has led to a rich body of works; cf. [6–8, 11–14, 19, 22, 30–34, 36, 38–41, 43, 44, 46–50]. (Not all of these references employ the quasi-sure notation in their presentation.)

The said connection to the Brownian sheet makes it clear that whenever $G \subseteq \Omega$ holds quasi-surely, then $G$ holds almost-surely as well. For a converse, it has been noted in [22, p. 165] that there are events that hold almost-surely and not quasi-surely. For instance, consider $G$ to be the collection of all continuous functions $f : [0,1] \to \mathbb{R}$ such that $f(1) \neq 0$. It is clear then that $G$ holds almost surely; equivalently, with probability one a Brownian motion $B$ satisfies $B(1) \neq 0$. On the other hand, $G$ does *not* hold quasi-surely. [This is equivalent to the statement that the Brownian sheet $W$ satisfies $W(t,1) = 0$ for some $t \geq 1$, which happens

with probability one since $t \mapsto W(t,1)$ is a Brownian motion, and hence is point-recurrent.]

I will say a few things in the final section of this paper about the aforementioned analytical methods and their potential-theoretic connections in turn. However, this paper is chiefly concerned with the aspects of quasi-sure analysis that are close in spirit to what I believe may be the general theme of this volume; namely, methods that are based on finite-dimensional processes, concentration, and Gaussian inequalities.

On a few occasions, $\Omega$ will denote the space of continuous functions $f : [0,1] \to \mathbb{R}^d$, and $W$ will denote $d$-dimensional two-parameter Brownian sheet, where $d \geq 1$. However, this should not cause any confusion.

## 2. A Strong Markov Property

The following is an infinite-dimensional strong Markov property of the Brownian sheet. It is not a particularly difficult result, but it is useful. In addition, this is a natural place to start our discussion.

Let $\mathfrak{F} := \{\mathfrak{F}(s);\ s \geq 0\}$ denote the filtration of $\sigma$-algebras defined as follows: For any $s \geq 0$, we first define $\mathfrak{F}_{00}(s)$ to be the $\sigma$-algebra generated by the random variables $\{W(r,t);\ r \in [0,s],\ t \geq 0\}$. To each $\mathfrak{F}_{00}(s)$ we can add all the P-null sets and call the resulting $\sigma$-algebra $\mathfrak{F}_0(s)$. Finally, we make this completed filtration right-continuous in the usual way; namely, we let $\mathfrak{F}(s) := \cap_{u>s} \mathfrak{F}_0(u)$.

**Theorem 2.1** (A Strong Markov Property). *If $S$ is a finite $\mathfrak{F}$-stopping time, then the process $t \mapsto W(S,t)$ is measurable with respect to $\mathfrak{F}(S)$. Moreover, the infinite-dimensional process $s \mapsto W(S+s,\bullet) - W(S,\bullet)$ is totally independent of $\mathfrak{F}(S)$, and has the same law as $W$.*

*Remark* 2.2. This is a simple consequence of J. B. Walsh's strong Markov property with respect to weak stopping points; cf. [48, Theorem 3.6] or [49, Theorem 1.6] for details.

*Proof.* Let $I(j;n)$ denote the half-open interval $[j2^{-n},(j+1)2^{-n})$, and for any fixed real number $r > 0$, define

$$S_{n,r} := \sum_{j=0}^{\lfloor 2^n r \rfloor} j2^{-n} \mathbf{1}_{I(j;n)}(S). \tag{2.1}$$

Since $S$ is an $\mathfrak{F}$-stopping time, so is $S_{n,r}$ for any fixed $n,r$; moreover, we have $S_{n,r} \leq S \wedge r$, and as $n \uparrow +\infty$, then $S_{n,r} \uparrow S\mathbf{1}_{\{S \leq r\}}$. Now

$$W(S_{n,r},t) = \sum_{j=0}^{\lfloor 2^n r \rfloor} W(j2^{-n},t) \mathbf{1}_{I(j;n)}(S). \tag{2.2}$$

In particular, for any $t_1,\ldots,t_k \geq 0$, the vector $(W(S_{n,r},t_i))_{1 \leq i \leq k}$ is $\mathfrak{F}(S_{n,r})$-measurable, which is another way to say that $W(S_{n,r},\bullet)$ is $\mathfrak{F}(S_{n,r})$-measurable. Since $\mathfrak{F}(S_{n,r}) \subseteq \mathfrak{F}(S)$, this shows that $W(S_{n,r},\bullet)$ is $\mathfrak{F}(S)$-measurable. Let $n \uparrow \infty$ and $r \uparrow \infty$ (along rationals), and use the path continuity of $W$ to see that $W(S,\bullet)$ is $\mathfrak{F}(S)$-measurable, as asserted. Suppose $\Phi$ is the random function

$$\Phi(u) := \prod_{i=1}^{m} \phi_i \Big( W(u+s_i,t_i) - W(u,t_i) \Big), \tag{2.3}$$

where $\phi_1, \ldots, \phi_m$ are bounded continuous functions, and $t_1, \ldots, t_m \geq 0$. Then, for any bounded $\mathfrak{F}(S_{n,r})$-measurable random variable $\xi$,

$$\mathrm{E}\left\{\Phi(S_{n,r}) \cdot \xi\right\} = \sum_{j=0}^{\lfloor 2^n r\rfloor} \mathrm{E}\Big\{\Phi(j2^{-n}) \cdot \xi \mathbf{1}_{I(n;j)}(S)\Big\}. \tag{2.4}$$

The term "$\xi$ times the indicator function" is $\mathfrak{F}(j2^{-n})$-measurable since $S$ is a stopping time. Therefore, the stationary independent-increments property of Brownian sheet implies that

$$\mathrm{E}\left\{\Phi(S_{n,r}) \cdot \xi\right\} = \mathrm{E}\{\Phi(0)\} \cdot \sum_{j=0}^{\lfloor 2^n r\rfloor} \mathrm{E}\left\{\xi \mathbf{1}_{I(n;j)}(S)\right\} = \mathrm{E}\{\Phi(0)\} \cdot \mathrm{E}\{\xi\}. \tag{2.5}$$

This shows that a.s.,

$$\mathrm{E}\left\{\Phi(S_{n,r}) \,\middle|\, \mathfrak{F}(S_{n,r})\right\} = \mathrm{E}\{\Phi(0)\}, \tag{2.6}$$

which is the desired strong Markov property in the case that $S \equiv S_{n,r}$. For the general case, we let $n, r \uparrow \infty$ along rationals, and use the fact that $\mathfrak{F}(S_{n,r}) \uparrow \mathfrak{F}(S)$, and that $\Phi(S_{n,r}) \to \Phi(S)$ boundedly, together with H. Föllmer's multiparameter version of Hunt's lemma ([20, Lemma 2.3]), to see that $\mathrm{E}\{\Phi(S) \,|\, \mathfrak{F}(S)\} = \mathrm{E}\{\Phi(0)\}$. This completes our proof. □

The above immediately yields a 0–1 law of the following form:

**Corollary 2.3** (A Zero-One Law)**.** *If $S$ is a finite $\mathfrak{F}$-stopping time, then the following $\sigma$-algebra is trivial:*

$$\mathfrak{A}(S) := \bigcap_{s \in \mathbb{Q}_+} \sigma\Big\{W(S+s, \bullet) - W(S, \bullet)\Big\}. \tag{2.7}$$

*Proof.* This follows from Theorem 2.1 and R. M. Blumenthal's 0-1 law (cf. [3]). However, we include an argument that we will need later on, but do not wish to repeat. Consider the infinite-dimensional process $X(s;\delta) := W(S+s, \bullet) - W(S+\delta, \bullet)$, as $s$ varies over $(\delta, \infty)$ and $\delta > 0$ is a fixed number. Thanks to Theorem 2.1 applied to the $\mathfrak{F}$-stopping times of the form $r + T$ (where $r$ is nonrandom), for any fixed $\delta > 0$, all $s_1, s_2, \ldots, s_n > \delta$, and for all bounded continuous functionals $\phi_1, \ldots, \phi_m$,

$$\mathrm{E}\left\{ \prod_{j=1}^m \phi_j\left(X(s_j;\delta)\right) \,\middle|\, \mathfrak{X}(\delta) \right\} = \mathrm{E}\left\{ \prod_{j=1}^m \phi_j\left(X(s_j;\delta)\right) \right\}, \qquad \text{a.s.,} \tag{2.8}$$

where $\mathfrak{X}(\delta)$ is the $\sigma$-algebra generated by $\{W(S+u, \bullet) - W(S, \bullet);\, u \in [0, \delta)\}$. Since $\cap_{\delta \in \mathbb{Q}_+} \mathfrak{X}(\delta) = \mathfrak{A}(S)$, we can let $\delta \downarrow 0$ and apply Hunt's lemma (cf. C. Dellacherie and P.-A. Meyer [16, Chapter V, p. 25]) to deduce that

$$\mathrm{E}\left\{ \prod_{j=1}^m \phi_j\left(X(s_j;0)\right) \,\middle|\, \mathfrak{A}(S) \right\} = \mathrm{E}\left\{ \prod_{j=1}^m \phi_j\left(X(s_j;0)\right) \right\}, \qquad \text{a.s.} \tag{2.9}$$

The restriction $s_i > \delta$ has been removed since $\delta$ has been allowed to go to zero while keeping the $s_i$'s fixed. Thus, (2.9) holds for all bounded continuous functionals $\phi_1, \ldots, \phi_m$ and all $s_1, \ldots, s_m > 0$. A monotone class argument shows that $\mathfrak{A}(S)$ is independent of itself, and is trivial as a result. □

## 3. A Law of the Iterated Logarithm

**Theorem 3.1** (G. J. Zimmerman [50, Theorem 3])**.** *Quasi-every $f \in \Omega$ satisfies the law of the iterated logarithm.*

*Remark* 3.2. For an analytic proof see [22, Theorem 4].

Equivalently, Zimmerman's LIL states that with probability one,

$$\limsup_{t \downarrow 0} \frac{U(s,t)}{\sqrt{2t \log\log\left(\frac{1}{t}\right)}} = 1, \qquad \forall s \in [0,1]. \tag{3.1}$$

Of course, the point is that the null set in question does not depend upon $s \in [0,1]$ (or for that matter upon $s \geq 0$ by scaling). It is easy to see that the preceding equation can be translated to the following statement about the Brownian sheet: With probability one,

$$\limsup_{t \downarrow 0} \frac{W(s,t)}{\sqrt{2st \log\log\left(\frac{1}{t}\right)}} = 1, \qquad \forall s \in [1,e]. \tag{3.2}$$

In the next two subsections we will prove this particular reformulation of Theorem 3.1. Before getting on with proofs, I would like to mention—without proof—the following theorem of [39]. Recall that a function $G : \mathbb{R}_+ \to \mathbb{R}$ is an *upper function* for a function $g : \mathbb{R}_+ \to \mathbb{R}$ if there exists $x_0 > 0$ such that for all $x \in [0, x_0]$, $g(x) \leq G(x)$.

**Theorem 3.3** (T. S. Mountford [39])**.** *An increasing function $t \mapsto \sqrt{t}\phi\left(\frac{1}{t}\right)$ is in the upper class of quasi-every function in $\Omega$ if and only if*

$$\int_4^\infty \phi^3(t) e^{-\phi^2(t)/2} \, \frac{dt}{t} < +\infty. \tag{3.3}$$

On the other hand, the upper class for almost all continuous paths has a different characterization that is described in the classic paper [18]; it states the following:

**Theorem 3.4** (P. Erdős [18])**.** *An increasing function $t \mapsto \sqrt{t}\phi\left(\frac{1}{t}\right)$ is in the upper class of almost every function in $\Omega$ if and only if*

$$\int_4^\infty \phi(t) e^{-\phi^2(t)/2} \, \frac{dt}{t} < +\infty. \tag{3.4}$$

To illustrate, for any $\alpha > 0$ define

$$\phi_\alpha(t) := \sqrt{2 \log\log\left(\frac{1}{t}\right) + \alpha \log\log\log\left(\frac{1}{t}\right)}, \qquad \forall t > 4, \tag{3.5}$$

and note that when $\alpha \in (3,5]$, $\phi_\alpha$ is an upper function for almost all continuous paths, but it is *not* an upper function for quasi-all of them.

We conclude this section by describing our proof of Theorem 3.1.

### 3.1. **Upper Bound.**

As in Khintchine's classical proof of the law of iterated logarithm, we begin by verifying the following half of (3.2): With probability one,

$$\limsup_{t \downarrow 0} \frac{W(s,t)}{\sqrt{2st \log\log\left(\frac{1}{t}\right)}} \leq 1, \qquad \forall s \in [1,e]. \tag{3.6}$$

To prove this, we will need an infinite-dimensional reflection principle that we state in the following abstract form.

**Lemma 3.5** (The Reflection Principle)**.** *If $B$ is a continuous Brownian motion in a separable Banach space $\mathbb{B}$, and if $\mathcal{N}$ is any seminorm on $\mathbb{B}$ that is compatible with the topology of $\mathbb{B}$, then for all $T, \lambda > 0$,*

$$\mathrm{P}\left\{ \sup_{t\in[0,T]} \mathcal{N}(B(t)) \geq \lambda \right\} \leq 2\mathrm{P}\left\{ \mathcal{N}(B(T)) \geq \lambda \right\}. \tag{3.7}$$

*Proof.* We follow the original ideas of D. André and P. Lévy that were developed for 1-dimensional Brownian motion.

Consider the stopping time

$$\sigma := \inf\left\{ t > 0 : \mathcal{N}(B(t)) \geq \lambda \right\}. \tag{3.8}$$

Since $\mathcal{N}$ is compatible with the topology of $\mathbb{B}$, and since $W$ is continuous, $\mathcal{N}(B(\sigma)) = \lambda$ on $\{\omega : \sigma(\omega) < +\infty\}$. Now

$$\begin{aligned} \mathrm{P}\left\{ \sup_{t\in[0,T]} \mathcal{N}(B(t)) \geq \lambda \right\} &= \mathrm{P}\left\{ \mathcal{N}(B(T)) \geq \lambda \right\} + \mathrm{P}\left\{ \sigma < T\ ,\ \mathcal{N}(B(T)) < \lambda \right\} \\ &= \mathrm{P}\left\{ \mathcal{N}(B(T)) \geq \lambda \right\} + \mathrm{P}\left\{ \sigma < T\ ,\ \mathcal{N}(B(T) - B(\sigma) + B(\sigma)) < \lambda \right\} \\ &= \mathrm{P}\left\{ \mathcal{N}(B(T)) \geq \lambda \right\} + \mathrm{P}\left\{ \sigma < T\ ,\ \mathcal{N}(-B(T) + 2B(\sigma)) < \lambda \right\}, \end{aligned} \tag{3.9}$$

thanks to the symmetry and independent-increments (i.e., the strong Markov; cf. Theorem 2.1) properties of $W$. But the seminorm property of $\mathcal{N}$ insures us of its subadditivity. Thus, on $\{\sigma < +\infty\}$ we have

$$\mathcal{N}(-B(T) + 2B(\sigma)) \geq 2\mathcal{N}(B(\sigma)) - \mathcal{N}(B(T)) = 2\lambda - \mathcal{N}(B(T)). \tag{3.10}$$

The previous two displays, used in conjunction, prove the result. □

*Proof of (3.6).* Fix $c, \theta > 1$, and consider the measurable events

$$\mathsf{F}_n := \left\{ \omega :\ \exists s \in [1, e],\ \sup_{0\leq t\leq \theta^{-n}} W(s,t) \geq \sqrt{2cs\theta^{-n}\log\log\theta^n} \right\}. \tag{3.11}$$

We can rewrite $\mathsf{F}_n$ as follows.

$$\mathsf{F}_n = \left\{ \omega : \sup_{0\leq t\leq \theta^{-n}} \mathcal{N}(B(t)) \geq \sqrt{2c\theta^{-n}\log\log\theta^n} \right\}, \tag{3.12}$$

where for all $f \in \Omega$, $\mathcal{N}$ is the seminorm

$$\mathcal{N}(f) = \sup_{1\leq s\leq e} \frac{f(s)}{\sqrt{s}}, \tag{3.13}$$

and $t \mapsto B(t)$ is the following Brownian motion in the Banach space $\mathbb{B}$ of continuous functions on $[1, e]$ endowed with its compact-open topology:

$$B(t)(s) = \frac{W(s,t)}{\sqrt{s}}. \tag{3.14}$$

It follows readily that all of the assumptions of the reflection principle are verified in the present context; cf. Lemma 3.5. Thus, the latter lemma implies that

$$\begin{aligned} \mathrm{P}\left(\mathsf{F}_n\right) &\leq 2\mathrm{P}\left\{ \sup_{1\leq s\leq e} \frac{W(s,\theta^{-n})}{\sqrt{s}} \geq \sqrt{2c\theta^{-n}\log\log\theta^n} \right\} \\ &= 2\mathrm{P}\left\{ \sup_{0\leq s\leq 1} O(s) \geq \sqrt{2c\log\log\theta^n} \right\}, \end{aligned} \tag{3.15}$$

where $O$ denotes a one-parameter Ornstein–Uhlenbeck process. C. Borell's inequality ([2]) shows that as $n \to \infty$, we have the estimate $\mathrm{P}\left\{\mathsf{F}_n\right\} \le n^{-c+o(1)}$. Since $c > 1$, $n \mapsto \mathrm{P}\{\mathsf{F}_n\}$ forms a summable sequence in $n$; thus, by the Borel–Cantelli lemma, with probability one, eventually $\mathsf{F}_n$ does not occur. Equivalently, with probability one,

$$\limsup_{n\to\infty} \frac{\sup_{0\le t\le \theta^{-n}} W(s,t)}{\sqrt{2s\theta^{-n}\log\log\theta^n}} \le \sqrt{c}, \qquad \forall s \in [1,e]. \tag{3.16}$$

Since $c, \theta > 1$ are arbitrary, monotonicity arguments yield (3.6). □

3.2. **Lower Bound.** Theorem 3.1 now follows once we show that

$$\limsup_{t\downarrow 0} \frac{W(s,t)}{\sqrt{2st\log\log\left(\frac{1}{t}\right)}} \ge 1, \qquad \forall s \in [1,e]. \tag{3.17}$$

*Proof of (3.17).* Fix four constants $\varepsilon, c \in (0,1)$, $\tau > 0$, and $\theta > 1$, and consider the events

$$\mathsf{E}_n := \left\{ \omega:\ \forall s \in [\tau, \tau(1+\varepsilon)],\ \frac{W(s,\theta^{-n}) - W(s,\theta^{-n-1})}{\sqrt{2cs(\theta^{-n}-\theta^{-n-1})\log\log\theta^n}} \ge 1 \right\}. \tag{3.18}$$

Evidently, independently of $\tau > 0$,

$$\mathrm{P}(\mathsf{E}_n) = \mathrm{P}\left\{ \inf_{1\le s\le 1+\varepsilon} \frac{B(s)}{\sqrt{s}} \ge \sqrt{2c\log\log\theta^n} \right\}, \tag{3.19}$$

where $B$ is a Brownian motion. Trivially, for any given $c' \in (c,1)$, and as $n \to \infty$,

$$\begin{aligned} \mathrm{P}\left(\mathsf{E}_n\right) &\ge \mathrm{P}\left\{ \inf_{1\le s\le 1+\varepsilon} B(s) \ge \sqrt{2c(1+\varepsilon)\log\log\theta^n} \right\} \\ &\ge \mathrm{P}\left\{ \sup_{1\le s\le 1+\varepsilon} |B(s)-B(1)| \le 1 \right\} \mathrm{P}\left\{ B(1) \ge \sqrt{2c'(1+\varepsilon)\log\log\theta^n} \right\} \\ &= n^{-c'(1+\varepsilon)+o(1)}. \end{aligned} \tag{3.20}$$

Now if we also insist that $c(1+\varepsilon) < 1$, then we can arrange things so that $c'(1+\varepsilon) < 1$. In this case, the independence of $\mathsf{E}_1, \mathsf{E}_2, \ldots$, used in conjunction with (3.20) and the Borel–Cantelli lemma, shows that infinitely many $\mathsf{E}_n$'s occur with probability one. Consequently, as long as $c(1+\varepsilon) < 1$, then outside one null set, the following holds simultaneously for all $s \in [\tau, \tau(1+\varepsilon)]$:

$$\begin{aligned} &\limsup_{n\to\infty} \frac{W(s,\theta^{-n})}{\sqrt{2cs(\theta^{-n}-\theta^{-n-1})\log\log\theta^n}} \\ &\ge 1 - \limsup_{n\to\infty} \frac{|W(s,\theta^{-n-1})|}{\sqrt{2cs(\theta^{-n}-\theta^{-n-1})\log\log\theta^n}}. \end{aligned} \tag{3.21}$$

The already-proven upper bound (cf. §3.1) implies that a.s., and simultaneously for all $s \in [\tau, \tau(1+\varepsilon)]$,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{|W(s,\theta^{-n-1})|}{\sqrt{2cs(\theta^{-n}-\theta^{-n-1})\log\log\theta^n}} \\ &= \frac{1}{\sqrt{\theta-1}} \limsup_{n\to\infty} \frac{|W(s,\theta^{-n-1})|}{\sqrt{2cs\theta^{-n-1}\log\log\theta^{n+1}}} \le \frac{1}{\sqrt{c(\theta-1)}}. \end{aligned} \tag{3.22}$$

Thus, by (3.21) and (3.22), a.s., and simultaneously for all $s \in [\tau, \tau(1+\varepsilon)]$,

$$
\begin{aligned}
&\limsup_{t\to 0} \frac{W(s,t)}{\sqrt{2cst\log\log\left(\frac{1}{t}\right)}} \ge \limsup_{n\to\infty} \frac{W(s,\theta^{-n})}{\sqrt{2cs\theta^{-n}\log\log\theta^n}} \\
&\ge \sqrt{1-\frac{1}{\theta}} \cdot \limsup_{n\to\infty} \frac{W(s,\theta^{-n})}{\sqrt{2cs(\theta^{-n}-\theta^{-n-1})\log\log\theta^n}} \ge \sqrt{1-\frac{1}{\theta}} - \sqrt{\frac{1}{c\theta}}.
\end{aligned} \tag{3.23}
$$

Since $\theta > 1$ is arbitrary, we can let $\theta \uparrow +\infty$ along a rational sequence to deduce that if $c(1+\varepsilon) > 1$, then almost surely,

$$
\limsup_{t\to 0} \frac{W(s,t)}{\sqrt{2cst\log\log\left(\frac{1}{t}\right)}} \ge 1, \qquad \forall s \in [\tau, \tau(1+\varepsilon)],\ \forall \tau \in \mathbb{Q}_+. \tag{3.24}
$$

Let $c \uparrow (1+\varepsilon)^{-1}$ along a rational sequence to see that

$$
\limsup_{t\to 0} \frac{W(s,t)}{\sqrt{2st\log\log\left(\frac{1}{t}\right)}} \ge \sqrt{\frac{1}{1+\varepsilon}}, \qquad \forall s \in [\tau, \tau(1+\varepsilon)],\ \forall \tau \in \mathbb{Q}_+. \tag{3.25}
$$

Equation (3.17) follows from this readily. □

## 4. J. B. Walsh's Proof of Theorem 3.1

The argument that was used to derive Theorem 3.1 (essentially due to G. J. Zimmerman) is quite natural, and has other uses in quasi-sure analysis as we shall see in the next section. I now wish to present a different derivation of Theorem 3.1—due to J. B. Walsh—that is elegant and short. It also has striking consequences on the "propagation of singularities" along the Brownian sheet. The main ingredient of Walsh's proof is the celebrated "section theorem" of [15] that, borrowing from the words of M. Sharpe, "is one of the prime achievements of [stochastic analysis]." See [45, p. 388].

4.1. **P.-A. Meyer's Section Theorem.** In order to describe a version of Meyer's section theorem that is suitable for our needs, we need to recall a few notions from the general theory of processes.

Let $(\Omega, \mathfrak{G}, \mathrm{P})$ denote a filtered probability space, where the filtration $\mathfrak{G} := (\mathfrak{G}_t)_{t\ge 0}$ satisfies the "usual conditions" of stochastic analysis, i.e., $\mathfrak{G}_0$ contains all the P-null sets, and $\mathfrak{G}_t = \cap_{r>t}\mathfrak{G}_r$. A stochastic process $\{X_t\}_{t\ge 0}$ is said to be *optional* if: (i) For all $t \ge 0$, $X_t$ is $\mathfrak{G}_t$-measurable; and (ii) $t \mapsto X_t(\omega)$ is right-continuous with left-limits for each $\omega \in \Omega$. The *optional* $\sigma$-algebra $\mathfrak{O}$ is the smallest $\sigma$-algebra of subsets of $[0,\infty)\times\Omega$ that renders optional processes measurable; i.e., $\mathfrak{O}$ is the $\sigma$-algebra generated by all sets of the form $\{(t,\omega) \in [0,\infty)\times\Omega :\ X_t(\omega) \in A\}$ where $A \subseteq \mathbb{R}$ is measurable and $X$ is an optional process. Finally, a stochastic set $\Gamma \subseteq [0,\infty)\times\Omega$ is *optional* if it is measurable with respect to $\mathfrak{O}$.

**Theorem 4.1** (P.-A. Meyer [15, Chapter IV, pp. 84–85]). *If $F$ is an optional set, then for every $\varepsilon > 0$, there exists a stopping time $T_\varepsilon : \Omega \to [0,\infty)$ such that $\mathrm{P}\{T_\varepsilon < +\infty\} \ge \mathrm{P}\{\Pi(F)\} - \varepsilon$, where $\Pi$ is the natural projection from $[0,\infty)\times\Omega$ onto $\Omega$.*

4.2. **J. B. Walsh's Proof.** For all $s \in [0,1]$ define

$$L_s := \limsup_{t \to 0^+} \frac{W(s,t)}{\sqrt{2t \log\log\left(\frac{1}{t}\right)}}. \tag{4.1}$$

The standard law of the iterated logarithm implies that for each fixed $s > 0$, $L_s = \sqrt{s}$, a.s. In particular, thanks to Fubini's theorem,

$$\mathrm{Leb}\left\{s > 0 : \; L_s \neq \sqrt{s}\right\} = 0, \qquad \text{a.s.}, \tag{4.2}$$

where Leb denotes Lebesgue's measure on $\mathbb{R}$. Our goal is to show that $\mathrm{P}\{\forall s > 0 : L_s = \sqrt{s}\} = 1$. Suppose to the contrary that $\mathrm{P}\{\exists s > 0 : \; L_s \neq \sqrt{s}\} > 0$. We will use the section theorem to obtain a contradiction. To do so, we need to meet the conditions of Theorem 4.1. Let $\mathfrak{G}_t := \mathfrak{F}(t)$ (the filtration of §2), that we recall satisfies the usual conditions. Let

$$F := \left\{(t,\omega) : \; L_t(\omega) \neq \sqrt{t}\right\}. \tag{4.3}$$

Since $s \mapsto W(s, \bullet)$ is continuous (in the space of continuous functions on $[0,1]$ endowed with compact-open topology), and since $\mathfrak{F}$ is generated by the latter process, the stochastic set $F$ is optional. By the section theorem (Theorem 4.1), there would then exist a finite $\mathfrak{F}$-stopping time $S$, such that $\mathrm{P}\{L_S \neq \sqrt{S}\} > 0$. (Technical remark: The process $s \mapsto W(s, \bullet)$ is not real-valued. However, by considering all processes of the form $\int W(s,t)\, \mu(dt)$ where $\mu$ is a linear combination of point masses, we can see that $\mathfrak{F}$ is generated by continuous real-valued processes, so that the section theorem can be applied as stated.) Without loss of generality, we can assume that there exists a $\delta > 0$ (fixed and nonrandom), such that $\mathrm{P}\{L_S < \sqrt{S} - \delta\} > 0$. (If for some $\delta > 0$, $\mathrm{P}\{L_S > \sqrt{S} + \delta\} > 0$, then a similar argument can be invoked to get a contradiction.)

Thanks to the strong Markov property (Theorem 2.1) and the usual LIL, for any $s > 0$, there exists a null set off of which,

$$L_{S+s} \leq L_S + \limsup_{t \to 0^+} \frac{W(S+s,t) - W(S,t)}{\sqrt{2t \log\log\left(\frac{1}{t}\right)}} = L_S + \sqrt{s}. \tag{4.4}$$

Therefore, for all $s > 0$ sufficiently small,

$$\mathrm{P}\left\{L_{S+s} < \sqrt{S+s}\right\} \geq \mathrm{P}\{L_{S+s} < \sqrt{S} + \sqrt{s} - \delta\} > 0. \tag{4.5}$$

We can integrate this $[ds]$ and use Fubini's theorem to deduce that with positive probability,

$$\mathrm{Leb}\left\{s > 0 : \; L_{S+s} < \sqrt{S+s}\right\} > 0, \tag{4.6}$$

which contradicts (4.2). □

## 5. Modulus of Continuity

A well-known result of P. Lévy ([35]) states that almost all $f \in \Omega$ have the following uniform modulus of continuity:

$$\limsup_{\varepsilon \to 0^+} \sup_{\substack{u,v \in [0,1]: \\ |u-v| \leq \varepsilon}} \frac{|f(u) - f(v)|}{\sqrt{2\varepsilon \log\left(\frac{1}{\varepsilon}\right)}} = 1. \tag{5.1}$$

In an elegant paper that popularized the subject of quasi-sure analysis, M. Fukushmia proved the following quasi-sure analogue.

**Theorem 5.1** (M. Fukushima [22, Theorem 3])**.** *Quasi-every $f \in \Omega$ has the uniform modulus of continuity described by (5.1).*

The argument of [22] involves infinite-dimensional analysis and Dirichlet form estimates. Instead of going that route, we follow a more classical route that has the advantage of providing us with a more delicate result. To explain this extension, we first recall that in their book ([9]), M. Csörgő and P. Révész have shown us that even if we replace the lim sup by a proper limit there, (5.1) holds for almost every function. By adapting their argument, we plan to prove the following refinement of Theorem 5.1.

**Theorem 5.2** (M. Fukushima [22, Theorem 3])**.** *Quasi-every $f \in \Omega$ satisfies (5.1) with* lim sup *replaced by a proper limit.*

*Proof.* Our goal is to show that with probability one,

$$\lim_{\varepsilon \to 0^+} \sup_{\substack{u,v \in [0,1]: \\ |u-v| \le \varepsilon}} \frac{|U(s,u) - U(s,v)|}{\sqrt{2\varepsilon |\log \varepsilon|}} = 1, \qquad \forall s \in [0,1]. \tag{5.2}$$

Instead, we will prove the following stronger result: Almost surely, as $\varepsilon \to 0^+$,

$$\sup_{\substack{u,v \in [0,1]: \\ |u-v| \le \varepsilon}} \frac{|U(s,u) - U(s,v)|}{\sqrt{2\varepsilon |\log \varepsilon|}} \longrightarrow 1, \tag{5.3}$$

*uniformly* for all $s \in [0,1]$. Clearly, this is equivalent to the following statement about the Brownian sheet that we propose to derive: With probability one,

$$\lim_{\varepsilon \to 0^+} \sup_{s \in [1,e]} \left| \sup_{\substack{u,v \in [0,1]: \\ |u-v| \le \varepsilon}} \frac{|W(s,u) - W(s,v)|}{\sqrt{2s\varepsilon |\log \varepsilon|}} - 1 \right| = 0. \tag{5.4}$$

5.1. **The Upper Bound.** Fix $0 < \theta < 1$, and define $\delta_n := n^2\theta^n$, and $\Theta_n := \{j\theta^n : 0 \le j \le \theta^{-n}\}$, and notice that

$$\mathrm{P}\left\{ \max_{\substack{u,v \in \Theta_n: \\ |u-v| \le \delta_n}} |W(s,u) - W(s,v)| \ge \sqrt{2s\delta_n\lambda} \right\} \le 2n^2 |\Theta_n| e^{-\lambda}, \tag{5.5}$$

since for every $v \in \Theta_n$, there are no more than $2n^2$ many $u \in \Theta_n$ such that $|u - v| \le \delta_n$. Let $\lambda := p \log(\delta_n^{-1})$ for a fixed $p > 1$, and appeal to our abstract form of reflection principle (Lemma 3.5) to deduce that for any $\tau > 0$,

$$\begin{aligned} &\mathrm{P}\left\{ \max_{s \in [0,\tau]} \max_{\substack{u,v \in \Theta_n: \\ |u-v| \le \delta_n}} |W(s,u) - W(s,v)| \ge \sqrt{2p\tau\delta_n \log(\delta_n^{-1})} \right\} \\ &\le 2\mathrm{P}\left\{ \max_{\substack{u,v \in \Theta_n: \\ |u-v| \le \delta_n}} |W(\tau,u) - W(\tau,v)| \ge \sqrt{2\tau p\delta_n \log(\delta_n^{-1})} \right\} \\ &\le \theta^{n(p-1)+o(n)}. \end{aligned} \tag{5.6}$$

On the other hand, by Kolmogorov's continuity theorem (cf. [28, Chapter 5, Exercise 2.5.1] for a suitable version of the latter theorem), for any integer $k \geq 1$, there exists a contant $A_k$ such that for all $r \in (0,1)$,

$$\mathrm{E}\left[\sup_{s\in[0,e]}\sup_{\substack{u,v\in[0,1]:\\ |u-v|\leq r}} |W(s,u)-W(s,v)|^k\right] \leq A_k r^{\frac{k}{2}}. \tag{5.7}$$

Thus, thanks to the triangle inequality, for any fixed $\eta > 0$ and $\ell \geq 0$,

$$\mathrm{P}\left\{\sup|W(s,u)-W(s,v)| \geq \psi_n\right\} \leq \theta^{n(p-1)+o(n)} + 2A_4 n^{-2}, \tag{5.8}$$

where the supremum is taken over all $s \in [1+\ell\eta, 1+(\ell+1)\eta]$ and $u,v \in [0,1]$ such that $|u-v| \leq \delta_n$, and $\psi_n := \sqrt{2p\{1+(\ell+1)\eta\}\delta_n \log\left(\delta_n^{-1}\right) + 2n\theta^{\frac{n}{2}}}$. The displayed probability is summable in $n$. But as $n \to \infty$, we have $\delta_n = (1+o(1))\delta_{n+1}$; moreover $p > 1$ is arbitrary. Therefore, the Borel–Cantelli lemma and monotonicity together show that with probability one

$$\limsup_{\varepsilon\to 0^+} \sup \frac{|W(s,u)-W(s,v)|}{\sqrt{2\varepsilon|\log\varepsilon|}} \leq \sqrt{1+(\ell+1)\eta}, \tag{5.9}$$

where the supremum is taken over all $s \in [1+\ell\eta, 1+(\ell+1)\eta]$ and $u,v \in [0,1]$ such that $|u-v| \leq \varepsilon$. For the $s$ in question, $1+(\ell+1)\eta \leq s(1+\eta)$. Since there are only finitely many integers $\ell$ to consider (namely, $0 \leq \ell \leq e/\eta$),

$$\limsup_{\varepsilon\to 0^+} \sup_{s\in[1,e]} \sup_{\substack{u,v\in[0,1]:\\ |u-v|\leq\varepsilon}} \frac{|W(s,u)-W(s,v)|}{\sqrt{2s\varepsilon|\log\varepsilon|}} \leq \sqrt{1+\eta}. \tag{5.10}$$

Let $\eta \downarrow 0$ along a rational sequence to deduce that a.s.,

$$\limsup_{\varepsilon\to 0^+} \sup_{s\in[1,e]} \sup_{\substack{u,v\in[0,1]:\\ |u-v|\leq\varepsilon}} \frac{|W(s,u)-W(s,v)|}{\sqrt{2s\varepsilon|\log\varepsilon|}} \leq 1. \tag{5.11}$$

This proves half of (5.4).

5.2. **The Lower Bound.** Notice that for any $p \in (0,1)$ fixed, all integers $n \geq 1$, and all $s > 0$,

$$\begin{aligned}
&\mathrm{P}\left\{\sup_{\substack{u,v\in[0,1]:\\ |u-v|\leq n^{-1}}} |W(s,u)-W(s,v)| \leq \sqrt{\frac{2ps}{n}\log n}\right\}\\
&\leq \mathrm{P}\left\{\max_{0\leq j\leq n-1}\left|W\left(s,\frac{j+1}{n}\right) - W\left(s,\frac{j}{n}\right)\right| \leq \sqrt{\frac{2ps}{n}\log n}\right\}\\
&= \left(1 - \mathrm{P}\left\{|\mathcal{N}| > \sqrt{2p\log n}\right\}\right)^n,
\end{aligned} \tag{5.12}$$

where $\mathcal{N}$ denotes a standard normal variable. Consequently, as $n \to \infty$,

$$\mathrm{P}\left\{\sup_{\substack{u,v\in[0,1]:\\ |u-v|\leq n^{-1}}} |W(s,u)-W(s,v)| \leq \sqrt{\frac{2ps}{n}\log n}\right\} \leq \exp\left(-n^{1-p+o(1)}\right), \tag{5.13}$$

where $o(1)$ goes to 0 uniformly in $s > 0$. Let

$$\sigma_n := \left\{1 + jn^{-4} : \; 0 \le j \le (n+1)^4\right\} \tag{5.14}$$

to see that as $n \to \infty$,

$$\sum_{n=1}^{\infty} \mathrm{P}\left\{ \min_{s\in\sigma_n} \sup_{\substack{u,v\in[0,1]:\\ |u-v|\le n^{-1}}} \frac{|W(s,u)-W(s,v)|}{\sqrt{s}} \le \sqrt{\frac{2p}{n}\log n} \right\} < +\infty. \tag{5.15}$$

Since $p \in (0,1)$ is arbitrary, by the Borel–Cantelli, and after applying another interpolation argument involving (5.7) and yet another monotonicity argument, we conclude that a.s.,

$$\liminf_{\varepsilon\to 0^+} \inf_{s\in[1,e]} \sup_{\substack{u,v\in[0,1]:\\ |u-v|\le\varepsilon}} \frac{|W(s,u)-W(s,v)|}{\sqrt{2s\varepsilon|\log\varepsilon|}} \ge 1. \tag{5.16}$$

Together with (5.11), this yields (5.4) whence (5.3). □

## 6. Nowhere-Differentiability

A classical result of R. E. A. C. Paley, N. Wiener and A. Zygmund ([42]) states that almost all continuous functions are nowhere-differentiable; see also [17]. This has been extended in various directions in [8], a consequence of which is the following; see [22, Theorem 2] for an analytical proof of most of this theorem.

**Theorem 6.1** (M. Csörgő and P. Révész [8, Theorem 3]). *Quasi-every $f \in \Omega$ is nowhere-differentiable.*

We mention—without proof—the following uniform modulus of nondifferentiability that is a two-parameter extension of the result of [10]: With probability one, as $\varepsilon \to 0^+$,

$$\inf_{t\in[0,T]} \sup_{u\in[0,\varepsilon]} \frac{|U(s,t+u)-U(s,t)|}{\sqrt{\varepsilon/|\log\varepsilon|}} \longrightarrow \frac{\pi}{\sqrt{8}}, \tag{6.1}$$

*uniformly* for all $s \in [0,1]$. In particular, the Csörgő–Révész modulus of nondifferentiability holds for quasi-all continuous functions.

In fact, M. Csörgő and P. Révész [8, Theorem 3] proved the much stronger theorem that the Brownian sheet is nowhere-differentiable along *any line in the plane.* Here we have specialized this result to the simpler case where the lines are parallel to one of the axes. A consequence of this more general theorem of [8] is that the level sets of the Brownian sheet a.s. do not contain straight-line segments. More recently, R. C. Dalang and T. S. Mountford have discovered a striking generalization of this fact:

**Theorem 6.2** (R. C. Dalang and T. S. Mountford [12]). *With probability one, the level curves of the Brownian sheet do not contain any curve that is differentiable somewhere.*

*Proof of Theorem 6.1.* Motivated by the analysis of [8], our strategy will be to show that if $T > 0$ is held fixed, then with probability one,

$$\inf_{s\in[0,1]} \limsup_{n\to\infty} \inf_{t\in[0,T]} \sup_{u\in[0,n^{-1}]} \frac{|U(s,t+u)-U(s,t)|}{n^{-1}} = +\infty. \tag{6.2}$$

This implies Theorem 6.1. With this in mind, let us first recall the following estimate: If $B$ denotes standard Brownian motion, then there exists a constant $p > 0$ such that for all $x > 0$, $-\log \mathrm{P}\left\{\sup_{0\le t\le 1}|B(t)| \le x\right\}$ is bounded below by $(px^2)^{-1}$; cf. K. L. Chung [5, Theorem 2]. Equivalently, for any $n \ge 1$ and all $t, a > 0$,

$$\mathrm{P}\left\{\sup_{u\in[0,n^{-1}]}|B(t+u)-B(t)| \le \frac{a}{n}\right\} \le \exp\left(-\frac{n}{pa^2}\right), \tag{6.3}$$

for the same constant $p$ whose valued does not depend on our choice of $(a,t,n)$. [In fact, the optimal choice is $p = 8\pi^{-2}$.] Consequently,

$$\begin{aligned}
&\mathrm{P}\left\{\inf_{t\in[0,T]}\sup_{u\in[0,n^{-1}]}|B(t+u)-B(t)| \le \frac{a}{n}\right\}\\
&\le \mathrm{P}\left\{\min_{0\le j\le n^4}\sup_{u\in[0,n^{-1}]}\left|B\left(jn^{-4}T+u\right)-B\left(jn^{-4}T\right)\right| \le \frac{2a}{n}\right\}\\
&\quad + \mathrm{P}\left\{\sup|B(u)-B(v)| \ge \frac{a}{2n}\right\}\\
&\le (1+n^4)\exp\left(-\frac{n}{4pa^2}\right) + \left(\frac{n}{2a}\right)^2 \mathrm{E}\left[\sup|B(u)-B(v)|^2\right],
\end{aligned} \tag{6.4}$$

where the last two suprema are over all $u, v \in [0,1]$ such that $|u-v| \le n^{-4}T$. By (5.7), the last expectation is seen to be no more than $A_2Tn^{-4}$. Consequently,

$$\sum_{n=1}^{\infty}\mathrm{P}\left\{\inf_{t\in[0,T]}\sup_{r\in[0,n^{-1}]}|B(t+r)-B(t)| \le \frac{a}{n}\right\} < +\infty. \tag{6.5}$$

An interpolation argument improves this condition to the following one for the Brownian sheet $W$:

$$\sum_{n=1}^{\infty}\mathrm{P}\left\{\inf_{s\in[0,e]}\inf_{t\in[0,T]}\sup_{r\in[0,n^{-1}]}|W(s,t+r)-W(s,t)| \le \frac{a}{n}\right\} < +\infty, \tag{6.6}$$

from which we can easily deduce (6.2). □

## 7. Quadratic Variation

We now come to the theorem that started much of the interest in quasi-sure analysis. Namely, D. Williams's quasi-sure refinement of the classical theorem of P. Lévy that states that for almost every continuous function $f$, at time $t$ the function $f$ has finite quadratic variation $t$; cf. the appendix of [38].

**Theorem 7.1** (D. Williams [15, Appendix])**.** *For each $n = 1, 2, \ldots$, let $0 = \pi_{0,n} < \pi_{1,n} < \cdots < \pi_{n,n} = 1$ denote a partition of $[0,1]$ such that $\sum_n \max_{1\le j\le n}(\pi_{j,n} - \pi_{j-1,n}) < +\infty$. Then quasi-every $f \in \Omega$ has the following property: For all $t \in [0,1]$,*

$$\lim_{n\to\infty}\sum_{j=1}^{n}\left|f(\pi_{j,n}t) - f(\pi_{j-1,n}t)\right|^2 = t. \tag{7.1}$$

*Proof.* In the interest of saving space, I will prove the slightly weaker statement that for each fixed $t > 0$, with probability one,

$$\lim_{n\to\infty} \sup_{1\le s\le e} \left| \sum_{j=1}^{n} \left| W(s, \pi_{j,n}t) - W(s, \pi_{j-1,n}t) \right|^2 - st \right| = 0. \tag{7.2}$$

The proof that is to follow can be enhanced, using similar ideas, to show that in fact the above holds outside a single null set, uniformly for all $t \in [0, 1]$, which yields the full statement of the theorem.

We define the following, all the time keeping $t \in [0, 1]$ fixed:

$$\theta_j(s) := \left| W(s, \pi_{j,n}t) - W(s, \pi_{j-1,n}t) \right|^2. \tag{7.3}$$

A few lines of calculations then show the existence of a universal constant $K_1$ such that $\mathrm{E}\{|\theta_j(s') - \theta_j(s)|^8\} \le K_1|s - s'|^4(\pi_{j,n} - \pi_{j-1,n})^4$. Thus, by the Kolmogorov continuity theorem ([28, Chapter 5, Exercise 2.5.1]), we can find a universal constant $K_2$ such that for all $\eta \in (0, 1)$, $j = 1, \ldots, n$, and $n = 1, 2, \ldots$,

$$\mathrm{E}\left\{ \max_{\substack{s,s'\in[1,e]:\\ |s-s'|\le\eta}} |\theta_j(s) - \theta_j(s')|^8 \right\} \le K_2\eta^3(\pi_{j,n} - \pi_{j-1,n})^4. \tag{7.4}$$

Now choose an equipartition $\mathbb{S}_n$ of $[1, e]$ with $\mathrm{mesh}(\mathbb{S}_n) \to 0$ at a rate to be described shortly, and note that for any fixed $\delta > 0$,

$$\mathrm{P}\left\{ \max_{s\in\mathbb{S}_n} |V_n(s, t)| \ge \delta \right\} \le \frac{\#(\mathbb{S}_n)}{\delta^2} \max_{s\in\mathbb{S}_n} \mathrm{Var}(V_n(s, t)), \tag{7.5}$$

where

$$V_n(s, t) := \sum_{j=1}^{n} |W(s, \pi_{j,n}t) - W(s, \pi_{j-1,n}t)|^2 - st. \tag{7.6}$$

On the other hand, $V_n(s, t)$ is a sum of $n$ i.i.d. random variables, and a simple computation yields a universal constant $K_3$ such that uniformly for all $s \le e$, $\mathrm{Var}(V_n(s, t)) \le K_3 \sum_{j=1}^{n}(\pi_{j,n} - \pi_{j-1,n})^2 \le K_3 \max_{1\le j\le n}(\pi_{j,n} - \pi_{j-1,n})) := K_3\|\Pi\|_n$. This yields

$$\mathrm{P}\left\{ \max_{s\in\mathbb{S}_n} |V_n(s, t)| \ge \frac{\delta}{2} \right\} \le \frac{K_3}{\delta^2} \#(\mathbb{S}_n)\|\Pi\|_n. \tag{7.7}$$

Thus, for all $n$ large,

$$\begin{aligned}
&\mathrm{P}\left\{ \sup_{s\in[1,e]} |V_n(s, t)| \ge \frac{\delta}{2} \right\} \\
&\le \frac{K_3}{\delta^2}\#(\mathbb{S}_n)\|\Pi\|_n + \mathrm{P}\left\{ \max_{\substack{s,s'\in[1,e]\\ |s-s'|\le\mathrm{mesh}(\mathbb{S}_n)}} \sum_{j=1}^{n} |\theta_j(s) - \theta_j(s')| \ge \frac{\delta}{4} \right\} \\
&\le \frac{K_3}{\delta^2}\#(\mathbb{S}_n)\|\Pi\|_n + K_2\left(\frac{4}{\delta}\right)^8 [\mathrm{mesh}(\mathbb{S}_n)]^3 \sum_{j=1}^{n}(\pi_{j,n} - \pi_{j-1,n})^4 \\
&\le \frac{K_3}{\delta^2}\#(\mathbb{S}_n)\|\Pi\|_n + K_2\left(\frac{4}{\delta}\right)^8 [\mathrm{mesh}(\mathbb{S}_n)]^3 \|\Pi\|_n^3,
\end{aligned} \tag{7.8}$$

thanks to (7.7). Since $\sum_n \|\Pi\|_n < \infty$ we can always choose $\mathbb{S}_n$ such that the left-hand side of the preceding display is summable (in $n$), and this proves (7.2). $\square$

## 8. W. S. Kendall's Theorem

Upto this point, we have adopted the viewpoint that the Brownian sheet (equivalently, the Ornstein–Uhlenbeck process) is a natural diffusion on the space of continuous functions. While this viewpoint provides us with a great deal of insight about the sheet, it completely ignores the effect of the geometry of the parameter space on the process. The following is a delightful example of the subtle effect of the geometry of the parameter space, and was discovered by W. S. Kendall.

**Theorem 8.1** (W. S. Kendall [26, Theorem 1.1]). *If $s, t > 0$ are arbitrary but fixed, then with probability one, the level curve of $W$ that goes through $(s, t)$ is totally disconnected.*

*Remark* 8.2. One can restate Theorem 8.1 in the following manner: Consider the level curve of $W$ that goes through $(s, t)$, and let

$$\begin{aligned}\Gamma(s,t) := \text{The connected component of } &\Big\{(u,v):\ W(u,v) = W(s,t)\Big\} \\ &\text{that contains } (s,t).\end{aligned} \tag{8.1}$$

By the continuity of $W$, this definition is perfectly well-defined, and Theorem 8.1 asserts that with probability one, $\Gamma(s,t) = \{(s,t)\}$. In other words, the excursion at the level $W(s,t)$ corresponding to the time-point $(s,t)$ is trivial. See the works of R. C. Dalang and J. B. Walsh [13, 14] who present very precise descriptions of the local structure of the excursions of the Brownian sheet.

*Remark* 8.3. The null set in question depends on the choice of $(s,t)$. Moreover, a little thought shows that Theorem 8.1 cannot hold a.s. simultaneously for all $(s,t)$ in any given open set. In this sense, this result is optimal.

The proof of Kendall's theorem rests on the following zero-one law; it is a two-parameter analogue of the infinite-dimensional zero-law described earlier in Corollary 2.3.

**Lemma 8.4** (S. Orey and W. E. Pruitt [41, p. 140]). *For any given $s, t > 0$, the following $\sigma$-algebra is trivial:*

$$\mathfrak{G}(s,t) := \bigcap_{\varepsilon \in \mathbb{Q}_+} \sigma\Big\{W(u,v) - W(s,t):\ |(s,t) - (u,v)| < \varepsilon\Big\}, \tag{8.2}$$

*where $|\cdots|$ denotes the $\ell^\infty$-norm on $\mathbb{R}^2$ for the sake of concreteness.*

*Proof.* **(Sketch)** This requires ideas that are very close to those introduced in the proof of Corollary 2.3. Thus, we will indicate only the essential differences between the two proofs.

We can think of Brownian sheet as the distribution function of white noise. Namely, let $\dot{\mathrm{W}}$ denote one-dimensional white noise spread over $\mathbb{R}^2$ and *define* the Brownian sheet $W$ as

$$W(u,v) := \dot{\mathrm{W}}\Big([0,u] \times [0,v]\Big), \qquad \forall u, v \geq 0. \tag{8.3}$$

[The process $\dot{\mathrm{W}}$ is a well-defined vector-valued random measure with values in $L^2(\mathrm{P})$; cf. [49, p. 283–285] and [28, Chapter 5, §1.3] for more details.] In this way, we can write

$$\mathfrak{G}(s,t) = \bigcap_{\varepsilon\in\mathbb{Q}_+} \sigma\left\{\dot{\mathrm{W}}\Big([0,s+\varepsilon]\times[0,t+\varepsilon]\setminus[0,s-\varepsilon]\times[0,t-\varepsilon]\Big)\right\}. \tag{8.4}$$

Now suppose that $R$ is a rectangle with sides parallel to the axes, and that $R$ does not intersect the annulus, $[0,s+\varepsilon]\times[0,t+\varepsilon]\setminus[0,s-\varepsilon]\times[0,t-\varepsilon]$. Then the elementary properties of white noise show us that $\dot{\mathrm{W}}(R)$ is independent of $\mathfrak{G}(s,t)$. To finish, consider a finite union of such $R$'s, and "take limits" in a manner similar to what we did in the proof of Corollary 2.3. □

*Proof of Theorem 8.1.* We may assume without loss of too much generality that $s=t=1$. Consider

$$\mathrm{B}(r) := \left\{(x,y)\in\mathbb{R}^2 :\ |(x,y)-(1,1)|\le r\right\}, \tag{8.5}$$

which is the $\ell^\infty$-ball of radius $r>0$ about $(1,1)\in\mathbb{R}^2$. Also let $\partial\mathrm{B}(r)$ denote its Euclidean boundary; this is the perimeter boundary of the square of side $2r$ centered at $(1,1)$. Let

$$\begin{aligned}\mathsf{J}(r) &:= \left\{\omega :\ W(1,1) > \sup_{(u,v)\in\partial\mathrm{B}(r)} W(u,v)\right\}\\ &= \left\{\omega :\ W(1,1) > \sup_{(u,v)\in\partial\mathrm{B}(r)} W\left(ru+1-r, rv+1-r\right)\right\}.\end{aligned} \tag{8.6}$$

Then the theorem follows at once from Lemma 8.4 and the following:

$$\liminf_{r\to0^+} \mathrm{P}\{\mathsf{J}(r)\} > 0. \tag{8.7}$$

For then it follows that with probability one, infinitely many of the $\mathsf{J}(n^{-1})$'s occur, and clearly this does the job. Therefore, it suffices to prove (8.7). By (8.3), we can write the following path decomposition: For any $r\in(0,1)$ fixed,

$$\begin{aligned}W(1-r+ur,1-r+vr) =& \sqrt{(1-r)r}\Big[X(u)+Y(v)\Big]\\ &+ rZ(u,v) - W(1-r,1-r),\end{aligned} \qquad \forall u,v\ge0, \tag{8.8}$$

where $X$ and $Y$ are standard Brownian motions, $Z$ is a standard Brownian sheet, and the three are independent from one another as well as from $W(1-r,1-r)$. Indeed, here are the formulas for $(X,Y,Z)$ in terms of the white noise $\dot{\mathrm{W}}$ of (8.3):

$$\begin{aligned}X(u) :=& \frac{1}{\sqrt{(1-r)r}}\dot{\mathrm{W}}\Big([1-r,1-r+ur]\times[0,1-r]\Big)\\ Y(v) :=& \frac{1}{\sqrt{(1-r)r}}\dot{\mathrm{W}}\Big([0,1-r]\times[1-r,1-r+vr]\Big)\\ Z(u,v) :=& \frac{1}{r}\dot{\mathrm{W}}\Big([1-r,1-r+ur]\times[1-r,1-r+vr]\Big).\end{aligned} \tag{8.9}$$

We only need these formulas to check the assertions about $(X,Y,Z)$, and this is only a matter of checking a few covariances. In light of the second equality in (8.6),

we have $\omega \in \mathsf{J}(r)$ if and only if

$$\begin{aligned}\sqrt{(1-r)r}\Big[X(1)+Y(1)\Big]+rZ(1,1)\\ > \sqrt{(1-r)r}\Big[X(u)+Y(v)\Big]+rZ(u,v),\end{aligned} \tag{8.10}$$

simultaneously for all $(u, v) \in \partial\mathrm{B}(1)$. But thanks to the sample function continuity of $Z$, the supremum of $|Z(u, v)|$ over all $(u, v) \in \partial\mathrm{B}(1)$ is bounded almost surely. Thus, we can divide the preceding display by $\sqrt{r}$ and let $r \to 0^+$ to see that

$$\lim_{r\to 0^+} \mathrm{P}\{\mathsf{J}(r)\} = \mathrm{P}\Big\{\forall (u,v)\in\partial\mathrm{B}(1):\ X(1)+Y(1) > X(u)+Y(v)\Big\}, \tag{8.11}$$

and it is easy to see that the latter probability is strictly positive. To see this, let $X_i := X(i)$, $Y_i := Y(i)$ $(i = 1, 2)$. Also let $X_2^+ := \sup_{0\le u\le 2} X(u)$, and $Y_2^+ := \sup_{0\le v\le 2} Y(v)$, and note that the latter probability is equal to

$$\begin{aligned}&\mathrm{P}\Big\{X_1+Y_1 > X_2+Y_2^+, X_1+Y_1 > X_2^+ + Y_2\Big\}\\ &\ge \mathrm{P}\Big\{X_1 - X_2 \ge 1, X_1 - X_2^+ \ge -1\Big\} \times \mathrm{P}\Big\{Y_2^+ - Y_1 \le 1, Y_2 - Y_1 \le -1\Big\},\end{aligned} \tag{8.12}$$

and this is easily seen to be positive. This verifies (8.7) and the result follows. □

## 9. Criterion for Hitting Points

Thus far, we have only touched upon results that hold (or do not hold) for quasi-every one-dimensional function. This in turn has led us to the one-dimensional Brownian sheet. Now we turn to results in higher dimensions. With this in mind, let $W$ denote the Brownian sheet in $d$ dimensions and $\mathfrak{m}_d$ the standard Lebesgue measure on the Lebesgue-measurable subsets of $\mathbb{R}^d$. Stated in terms of Wiener measure, this yields the following.

We begin with the classical fact that $d$-dimensional Brownian can hit points if and only if $d = 1$.

**Theorem 9.1** (P. Lévy)**.** *The following are equivalent: Given any $x \in \mathbb{R}^d$,*

(i) *Almost every continuous $f : [0, 1] \to \mathbb{R}^d$ avoids $\{x\}$; i.e., $x \notin f([0, 1])$.*
(ii) *Almost every continuous $f : [0, 1] \to \mathbb{R}^d$ has a Lebesgue-null range; i.e., $\mathfrak{m}_d(f([0, 1])) = 0$.*
(iii) *$d \ge 2$.*

For the above conditions (i) and (ii) to hold for quasi-every function $f$, one needs the stronger condition that $d \ge 4$. Indeed, S. Orey and W. E. Pruitt have proven the following result.

**Theorem 9.2** (S. Orey and W. E. Pruitt [41, Theorems 3.3 and 3.4])**.** *The following are equivalent: Given any fixed $x \in \mathbb{R}^d$:*

(i) *With probability one, $d$-dimensional Brownian sheet does not hit $\{x\}$.*
(ii) *The random set $W(\mathbb{R}^2_+)$ has zero $d$-dimensional Lebesgue measure.*
(iii) *$d \ge 4$.*
(iv) *Quasi-every $d$-dimensional continuous function avoids $x$.*

Let $U(s, t) := e^{-s/2}W(e^s, t)$ as before, and note that (iv) is equivalent to the following:

$$\mathrm{P}\left\{\exists s\in[0,1]:\ \text{ for some } t>0,\ U(s,t)=x\right\} = 0, \qquad \forall x\in\mathbb{R}^d. \tag{9.1}$$

On the other hand, by the Cameron–Martin formula, the law of $\{U(s,t);\, s,t \in [0,1]\}$ is mutually absolutely continuous with respect to the law of $\{W(s,t);\, s \in [1,e], t \in [0,1]\}$; cf. [4]. Therefore, (iv)$\Leftrightarrow$(i), and we only need to prove the equivalence of (i)–(iii). I will describe most of this proof in three steps.

*Proof of (i)$\Rightarrow$(ii).* By Fubini's theorem,

$$\mathrm{E}\left\{\mathfrak{m}_d\Big(W([0,1]^2)\Big)\right\} = \int_{\mathbb{R}^d} \mathrm{P}\left\{x \in W([0,1]^2)\right\}\, dx = 0, \tag{9.2}$$

thanks to (i). Scaling then shows that with probability one, $\mathfrak{m}_d(W(\mathbb{R}^2_+)) = 0$. $\square$

*Proof of (ii)$\Rightarrow$(iii).* We will use the Fourier-analytical ideas of [25], and prove that if (iii) fails, then so will (ii). Thus, let us assume that $d \le 3$, and consider the *occupation (or sojourn) measure*,

$$\sigma(A) := \int_0^\infty \int_0^\infty e^{-s-t}\mathbf{1}_A\left(W(s,t)\right)\, ds\, dt. \tag{9.3}$$

Its Fourier transform is given by

$$\widehat{\sigma}(\xi) = \int_0^\infty \int_0^\infty e^{-s-t} e^{i\xi\cdot W(s,t)}\, ds\, dt, \qquad \forall \xi \in \mathbb{R}^d. \tag{9.4}$$

Our strategy is to show that with probability one, $\widehat{\sigma} \in L^2(\mathbb{R}^d)$. If so, then by the Plancherel theorem, $\sigma$ is a.s. absolutely continuous with respect to $\mathfrak{m}_d$, and $\left(\frac{d\sigma}{d\mathfrak{m}_d}\right) \in L^2(\mathbb{R}^d)$ almost surely. But the fact that $\sigma\left(W(\mathbb{R}^2_+)\right) = 1$ implies that

$$\mathfrak{m}_d(W(\mathbb{R}^2_+)) = \int_{W(\mathbb{R}^2_+)} \frac{d\sigma(\xi)}{d\mathfrak{m}_d}\, \mathfrak{m}_d(d\xi) = 1. \tag{9.5}$$

Thus, (ii)$\Rightarrow$(iii) follows once we show that $\mathrm{E}\{\|\widehat{\sigma}\|^2_{L^2(\mathbb{R}^d)}\} < +\infty$. The latter expectation is equal to the following:

$$\begin{aligned}
&\mathrm{E}\left\{\|\widehat{\sigma}\|^2_{L^2(\mathbb{R}^d)}\right\} \\
&= \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^d} e^{-s_1-s_2-t_1-t_2}\mathrm{E}\left\{e^{i\xi\cdot[W(s_1,s_2)-W(t_1,t_2)]}\right\}\, d\xi\, ds\, dt \\
&= \int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^d} e^{-s_1-s_2-t_1-t_2}\exp\left(-\frac{|\xi|^2\tau^2(s,t)}{2}\right)\, d\xi\, ds\, dt,
\end{aligned} \tag{9.6}$$

where $\tau^2(s,t) := \mathrm{Var}[W(s_1,s_2) - W(t_1,t_2)]$ is the $\mathfrak{m}_d$-measure of the set difference between the rectangles $[0,s_1]\times[0,s_2]$ and $[0,t_1]\times[0,t_2]$. A picture will convince you that no matter how the two said rectangles are situated, we always have the bound, $\tau^2(s,t) \ge (s_2 \wedge t_2)|s_1 - t_1| + (s_1 \wedge t_1)|s_2 - t_2|$. Thus,

$$\begin{aligned}
&\mathrm{E}\left\{\|\widehat{\sigma}\|^2_{L^2(\mathbb{R}^d)}\right\} \\
&\le 4\iint_{0\le s_1\le t_1}\iint_{0\le s_2\le t_2}\int_{\mathbb{R}^d} e^{-s_1-s_2-t_1-t_2} \\
&\quad\times \exp\left(-\frac{|\xi|^2\left[s_2(t_1-s_1)+s_1(t_2-s_2)\right]}{2}\right)\, d\xi\, ds\, dt \\
&= 4\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^2_+}\int_{\mathbb{R}^d} e^{-2s_1-2s_2-t_1-t_2}\exp\left(-\frac{|\xi|^2\left[s_2t_1+s_1t_2\right]}{2}\right)\, d\xi\, ds\, dt.
\end{aligned} \tag{9.7}$$

We can integrate $[dt]$ to deduce that $\mathrm{E}\{\|\widehat{\sigma}\|^2_{L^2(\mathbb{R}^d)}\} \le 4\int_{\mathbb{R}^d} \mathcal{Q}(\xi)\, d\xi$, where

$$\mathcal{Q}(\xi) := \int_{\mathbb{R}^2_+} e^{-2(s_1+s_2)} \left(1 + \frac{|\xi|^2 s_1}{2}\right)^{-1} \cdot \left(1 + \frac{|\xi|^2 s_2}{2}\right)^{-1} ds. \tag{9.8}$$

Evidently, $\mathcal{Q}(\xi)$ is bounded, and is $O(|\xi|^{-4})$ as $|\xi| \to \infty$. Therefore, whenever $d < 4$, then $\mathcal{Q} \in L^1(\mathbb{R}^d)$, and so $\mathrm{E}\{\|\widehat{\sigma}\|^2_{L^2(\mathbb{R}^d)}\} < +\infty$, as asserted. □

*Partial Proof of (iii)⇒(i).* This is the most interesting, as well as difficult, part of Theorem 9.2, and I will present a proof that is valid in the "supercritical" regime $d \ge 5$. When $d = 4$, the known proofs are much longer and not included here. Clearly, it suffices to show that whenever $0 < a < b$, $\mathrm{P}\{x \in W([a,b]^2)\} = 0$. Without loss of too much generality, we will do this for $a = 1$ and $b = 2$.

Let us fix $\varepsilon \in (0,1)$ and an integer $n \ge 1$, and consider the covering $[1,2]^2 = \cup_{i,j=0}^n I_{i,j}$, where $I_{i,j} := [1 + (i/n), 1 + (i+1)/n] \times [1 + (j/n), 1 + (j+1)/n]$. Now if there exists $(s,t) \in I_{i,j}$ such that $|W(s,t) - x| \le \varepsilon$, then

$$\begin{aligned} &\left| W\left(1 + \frac{i}{n}, 1 + \frac{j}{n}\right) - x \right| \\ &\quad \le \varepsilon + \sup_{0 \le u,v \le \frac{1}{n}} \left| W\left(1 + \frac{i}{n}, 1 + \frac{j}{n}\right) - W\left(1 + \frac{i}{n} + u, 1 + \frac{j}{n} + v\right) \right| \\ &\quad := \varepsilon + \delta_{i,j;n}. \end{aligned} \tag{9.9}$$

Because of the white noise representation (8.3) of $W$, $\delta_{i,j;n}$ is independent of $W(1 + in^{-1}, 1 + jn^{-1})$. Moreover, the probability density of $W(1 + in^{-1}, 1 + jn^{-1})$ is uniformly bounded above by one. Therefore,

$$\begin{aligned} \mathrm{P}\left\{\exists (s,t) \in I_{i,j} :\ |W(s,t) - x| \le \varepsilon\right\} &\le C_d \mathrm{E}\left[(\varepsilon + \delta_{i,j;n})^d\right] \\ &\le 2^d C_d \mathrm{E}\left[\varepsilon^d + \delta_{i,j;n}^d\right], \end{aligned} \tag{9.10}$$

where $C_d$ denotes the volume of the unit ball in $\mathbb{R}^d$. On the other hand, by the white noise representation of $W$,

$$\begin{aligned} &W\left(1 + \frac{i}{n} + u, 1 + \frac{j}{n} + v\right) - W\left(1 + \frac{i}{n}, 1 + \frac{j}{n}\right) \\ &\quad = \sqrt{1 + \frac{j}{n}}\, B(u) + \sqrt{1 + \frac{i}{n}}\, B'(v) + Z(u,v), \end{aligned} \tag{9.11}$$

where $B$, $B'$, and $Z$ are independent, $B$ and $B'$ are Brownian motions, and $Z$ is a Brownian sheet. Consequently, we can take absolute values and maximize over $u, v \le n^{-1}$ to see that given $0 \le i, j \le n^{-1}$,

$$\delta_{i,j;n} \le \sqrt{2} \sup_{u \in [0,1/n]} |B(u)| + \sqrt{2} \sup_{v \in [0,1/n]} |B'(v)| + \sup_{u,v \in [0,1/n]} |Z(u,v)|. \tag{9.12}$$

This and scaling show the existence of a constant $K_d$ such that $\mathrm{E}[\delta^d_{i,j;n}] \le K_d n^{-d/2}$. Thus, according to (9.10),

$$\mathrm{P}\left\{\exists (s,t) \in I_{i,j} :\ |W(s,t) - x| \le \varepsilon\right\} \le 2^d C_d K_d \left[\varepsilon^d + n^{-d/2}\right]. \tag{9.13}$$

We can sum this over all $0 \le i, j \le n$ to see that

$$\mathrm{P}\left\{\exists (s,t) \in [1,2]^2 :\ |W(s,t) - x| \le \varepsilon\right\} \le 2^d C_d K_d (n+1)^2 \left[\varepsilon^d + n^{-d/2}\right]. \tag{9.14}$$

Because this is valid for all $n \ge 1$, we can choose $n := \lfloor \varepsilon^{-2} \rfloor$, and deduce that

$$\sup_{x\in\mathbb{R}^d} \mathrm{P}\left\{\exists (s,t) \in [1,2]^2 : \ |W(s,t) - x| \le \varepsilon\right\} = O\left(\varepsilon^{d-4}\right), \qquad (\varepsilon \to 0). \tag{9.15}$$

In particular, if $d \ge 5$, then with probability one, $x \notin W([1,2]^2)$, as claimed. □

## 10. The O-U Process on Wiener Space and Two Open Problems

In this section, I very briefly sketch the connection between the process $Y$ and symmetric forms. This is the starting point for the introduction of the methods of potential theory; an area which is not the focus of the present article. I then conclude this discussion by presenting two of my favorite open problems in this general area.

We have seen already that the Ornstein–Uhlenbeck process $Y$ of (1.3) is a stationary diffusion on $\Omega$ (the space of all real continuous functions on $[0,1]$) whose stationary measure is Wiener's measure. Moreover, in the usual notation of Markov processs, we have the following for all continuous $\phi : \Omega \to \mathbb{R}_+$, $s > 0$, and $x \in \Omega$:

$$\begin{aligned} T_s\phi(x) &:= \mathrm{E}\big[\phi(Y_s) \,\big|\, Y_0 = x\big] = \mathrm{E}\left[\phi\Big(e^{-s/2}W(e^s,\bullet)\Big) \,\Big|\, W(1,\bullet) = x\right] \\ &= \mathrm{E}\left[\phi\Big(e^{-s/2}\Big[W(e^s,\bullet) - W(1,\bullet)\Big] + e^{-s/2}x\Big)\right] \\ &= \mathrm{E}\left[\phi\Big(e^{-s/2}\Big[W(e^s - 1,\bullet)\Big] + e^{-s/2}x\Big)\right] \\ &= \mathrm{E}\left[\phi\Big(\sqrt{1-e^{-s}}W(1,\bullet) + e^{-s/2}x\Big)\right]. \end{aligned} \tag{10.1}$$

On the other hand, $W(1,\bullet)$ is just a Brownian motion, and this leads to the following.

**Lemma 10.1** (Mehler's Formula)**.** *If $\mu$ denote the Wiener measure on the classical Wiener space $(\Omega, \mathcal{B}(\Omega))$, then the transition semigroup of the diffusion $Y$ are given by the operator-formula: For all $s > 0$ and all continuous functions $x \in \Omega$,*

$$T_s f(x) = \int_\Omega f\left(\sqrt{1-e^{-s}}y + e^{-s/2}x\right)\, \mu(dy). \tag{10.2}$$

It is also a simple matter to check that $T_s$ is a symmetric semigroup on the classical Wiener space; i.e., that $\langle g, T_s f\rangle_\Omega = \langle T_s g, f\rangle_\Omega$, where $\langle u, v\rangle_\Omega$ is the covariance form, $\int_\Omega uv\, d\mu$.

Thus, the standard theory of symmetric Markov processes constructs a Dirichlet form $\mathcal{E}$ for $Y$ killed at an exponential rate that is formally defined as follows:

$$\mathcal{E}(f,g) := \lim_{s\to 0^+} \left\langle s^{-1}(f - T_s f), g\right\rangle_\Omega + \langle f, g\rangle_\Omega; \tag{10.3}$$

cf. [21, (1.3.15) and Theorem 1.4]. Now given any open set $G \subseteq \Omega$ one has the following identity; it relates the capacity of §1 to the Dirichlet forms of [21]:

$$\mathrm{Cap}(G) = \inf\Big\{\mathcal{E}(f,f);\ f \in \mathrm{dom}(\mathcal{E}),\ f \ge 1\ \mu\text{-a.e. on } G\Big\}, \tag{10.4}$$

where $\mathrm{dom}(\mathcal{E})$ denotes the domain of $\mathcal{E}$; i.e., all $f \in L^2(\Omega)$ such that $\mathcal{E}(f,f) < +\infty$. Furthermore, for a general set $A \subseteq \Omega$, $\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(G) :\ A \subseteq G \text{ open},\}$. The latter remarks are proved in [22, p. 164], and are another starting point for the analytic treatment of many of the quasi-sure results within the references.

As promised earlier, we conclude this paper by presenting two open problems:

**OP-1** P. Malliavin has introduced a parametric family of Gaussian capacities, one of which is the Cap of this and the first section. Is there a "truly probabilistic" description (i.e., one involving concrete random processes) of all of these capacities? If so, do the quasi-sure results of this paper continue to holds if Cap is everywhere replaced by any and all of the said capacities? For related results, see the results of M. Takeda ([47]).

**OP-2** One of the outstanding open problems of the geometry of Brownian sheet is the following: Let $\mathcal{L}$ denote the complement of the zero-set of an $N$-parameter Brownian sheet. Does the complement of $\mathcal{L}$ have an infinite connected component? When $N = 2$, this was answered in the negative by W. S. Kendall ([26]), but the general question remains open.

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